Changing conceptions of algebra:

What's <u>really</u> new in new curricula?

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Fundamentally new mathematics curricula serve students well when they provide everyone with richer and more accessible introductions to a wide range of mathematical content. But new curricula also serve teachers well when they lead us to examine and reflect upon what and how we teach. When these curricula enter our working lives and conversations, we are often forced to try to question exactly what is “new” about them and how this “newness” may affect our students’ learning. To address this issue and, we hope, to support further reflection and discussion, we take a closer and more careful look at what is “new” in one middle school curriculum's approach to algebra. The curriculum we examine is the Connected Mathematics Project [CMP] (Lappan et al., 1998) (particularly the 8th grade units), but the issue of what is new in algebra is relevant to many other innovative middle school curricula as well.

Identifying Differences between Older and Newer Conceptions of Algebra

What does it mean for students to develop solid understandings of algebraic ideas and concepts in middle school? Indeed, what do we mean when we say "algebraic ideas and concepts"? We have been pursuing these questions for some time, along with middle school, high school, and district colleagues, many of whom have had extensive experience teaching the 7th and 8th grade CMP materials. These discussions helped us generate six dimensions that captured much of the difference between traditional conceptions (and teaching) of algebra and the CMP introduction to algebra. Eventually, we compiled these dimensions into Table 1. Although students encounter algebraic concepts throughout the CMP program (grades 6–8), a substantial part of the 8th grade material addresses algebraic themes and content, so we decided to compare that year's curriculum with the content of traditional Algebra I.

** Insert Table 1 about here **

We explicitly avoided falling into the trap of calling one side of Table 1 “good” and the other “bad,” as many current discussions of algebra reform have tended to do. As former teachers, we know that there were plenty of outstanding curricula and ways of teaching prior to the introduction of the NCTM Standards in 1989. More importantly, we believe that broad and general claims of “good” and “bad” curricula are not helpful because they stop our thinking and reflection just when it should start. Instead of merely labeling a curriculum or teaching approach as "good" or "bad", we prefer to ask questions such as, "What about the curriculum is good (and bad)?", and "How does the curriculum affect student learning?" To move our own discussion of algebra curricula in a more productive direction, we developed Table 1 to represent the changing nature of school algebra. In the balance of the article, we illustrate these changes in relation to specific features of the CMP curriculum.

Fundamental Objects of Study

First, the mathematical objects we study in algebra have changed. Prior to the publication of the Standards-inspired curricula, algebra was almost exclusively the study of equations and symbolic expressions. Work on these objects produced solutions to equations and equivalent expressions via various manipulations. In fact, if one were to thumb through a traditional Algebra I textbook, one would find very few pages that do not contain symbolic expressions. The first chapter of those texts is often entitled “Expressions and Equations.” Instructions to teachers indicate explicitly that the course is primarily concerned with the development of familiarity and fluency with symbolic expressions and equations. For example, the following is from the “Foreword to the Teacher” in a popular textbook series (which the first author used for many years):

> The unifying theme is the concept of an expression [emphasis in original]. For increasingly complex expressions, students do these three things:
> 1. Write an expression representing a variable quantity in some real-world situation,
> 2. Find the value of the expression when *x* is known,
> 3. Find *x* when the value of the expression is known. (Foerster, 1990)

Newer conceptions of algebra present functional relationships as the fundamental object. In contrast to equations, functional relationships specify how one quantity changes in relation to changes in a second quantity. They are accessible precursors to the mathematical concept of function. When variable symbols are introduced in the study of functional relationships, they clearly represent true variables: numbers that vary over some numerical domain. This meaning of variable is quite different than the "unknown number" meaning that is carried by equations and their solutions (Usiskin, 1988). In other words, it is often the case that the symbols in equations that we refer to as "variables" do not vary like "true variables." For students, "variables" in equations are more like "numbers whose values we don't know yet."

In CMP and numerous other middle and high school curricula, functional relationships are presented in contextual problems that describe some realistic or fanciful situation. Often, the situation itself also contains a table of numerical values of the two quantities, a graph of the relationship, or an expression symbolizing the relationship. (See Figure 1 for a typical example of such a CMP problem.)

** Insert Figure 1 about here **

This emphasis on multiple representations of functional relationships can be seen in the following excerpt from the publication, Getting to Know the Connected Mathematics Program:

> CMP Algebra Goals -- By the End of the 8th Grade in CMP Most Students Should Be Able to:
>
> - Recognize situations in which important problems and decisions involve relations among quantitative variables -- one variable changing over time or several variables changing in response to each other.
> - Use numerical tables, graphs, symbolic expressions, and verbal descriptions to describe and predict the patterns of change in variables.
> - Recognize (in various representational forms) the patterns of change associated with linear, exponential, and quadratic functions.
> - Use numeric, graphic, and symbolic strategies to solve common problems involving linear, exponential, and quadratic functions. (Lappan et al., 1996)

Typical Problems

One implication of this shift in the fundamental objects of study has been a corresponding change in what typical problems look like. For some time, there have been essentially two types of problems in Algebra I: (1) symbolic expressions (or equations) that students were directed to factor, simplify, multiply, expand, or solve, and (2) word problems. Solutions to word problems typically involved generating and solving an equation and interpreting the numerical answer in the problem context. Though word problems are distributed throughout the Algebra I texts, students spend much more time with symbolic manipulations.

By contrast, almost all problems in 8th grade CMP materials are word problems. But CMP word problems differ from Algebra I word problems in several ways. They do not fit into the common Algebra I problem categories (e.g., age problems, coin problems, consecutive number problems); they often present situations that are either familiar or experientially real to students; and most are accompanied by tables, graphs, and/or symbolic expressions. Students are directed to do a variety of things with the problem, including "explain", "predict", "describe", "sketch", "investigate", and "explore." Students are asked to generate symbolic expressions and equations, but along with (and arguably less often than) other representations, especially tables and graphs.

Typical Solution Methods

Since typical problems are different, typical solution methods also differ. In Algebra I problems where students are expected to simplify, expand, factor, or solve, solution methods involve completing the correct manipulations in the correct order. Efficiency and fluency are valued attributes of students' work with such problems. Once they master the basic procedures, students are expected to develop shortcuts and recognize special cases. These basic and streamlined manipulations are what college mathematics professors are referring to when they say, "The rest is just algebra."

Typical solution methods to CMP algebra problems are quite different. They involve working with and interpreting verbal statements and/or the accompanying representations. Often students are asked to create additional representations for the embedded functional relationship and to write an explanation for numerical solutions.

For example, in a typical problem from the Moving Straight Ahead unit - ACE #4, p. 10 (Lappan et al., 1998), students are provided with a table showing the distance that a tour van traveled while moving at a constant speed (see Figure 2). In order to complete this problem, students must construct and interpret both tabular and graphical representations of the given data. The clarity, logic, and thoroughness of students' explanations are valued attributes of the solutions to this problem and ones like it — a very different list of attributes then the efficiency and fluency of symbolic manipulation!

** Insert Figure 2 about here **

Role of Practice

The role of skill practice is another dimension in which older and newer conceptions of algebra differ. In traditional Algebra I curricula, practice plays a very important role in students' learning of the content. A day's lesson typically involves the introduction of a new solution procedure or the modification of an existing procedure. Teachers usually present and explain worked-out examples for students to observe. Students then learn to use this new material through practice on a number of short and quite similar problems. Homework assignments provide additional practice on the problems covered in that day's lesson. In addition, homework may include problems from earlier material that gets repeatedly "cycled" for additional practice. Practice is considered a useful, if not indispensable, way to develop mastery of symbolic procedures. Indeed, structuring the curriculum around a set of procedures makes it much easier to organize students' practice.

In newer conceptions of algebra, practice plays a more limited role. CMP problems tend to be longer and have more parts, which means that students work on fewer problems in the course of a lesson (both in class and on homework). Also, similarities between problems are less salient. Even when several problems in a unit present and illustrate the same concept(s), their similarities are less apparent due to the varied ways in which verbal, graphical, tabular, and symbolic representations are used. This diversification makes it more difficult to conceive of practice with CMP problems because they vary, one to another, along so many dimensions. Classifying problems according to embedded functional relationship (e.g., linear, exponential, or quadratic) falls well short of specifying what students should do to solve them. On the other hand, CMP students receive a great deal of practice asking and answering a set of common mathematical questions, such as "What is going on this situation," "Does it make sense," and "What is varying in this situation?"

Role of Technology

Changing notions of algebra are also reflected in the role of technology. Calculators are used and explicitly called for in some (though not all) traditional algebra courses, including Algebra I. However, their use is usually balanced with paper-and-pencil computation, which is typically more highly valued. Paper-and-pencil calculation is viewed as crucial to the development of fluency with symbol manipulation procedures. Calculator use may hamper students' efforts to achieve fluency especially when the calculator can produce the solution instantly, with little (or no) work from the student. For example, calculators that can factor symbolic expressions may not be a sensible tool for students in a lesson focused on mastery of factoring procedures. Similarly, if students are expected to graph a linear equation by hand, graphing calculators that can generate such the graph instantly from the equation may not be appropriate. In Algebra I, calculators are generally valued for computing numerical values, such as products, sums, quotients, powers, and square roots, so that students can concentrate on other aspects of the problem.

A much wider use of technology is encouraged with CMP materials. The curriculum makes two quite strong commitments to technology:

> (1) Students will have access to calculators at all times... In the 7th and 8th Grades we assume that students will have graphing calculators with table and statistical-display capability; and, (2) computer software will be provided with the curriculum that students will be able to use in tandem with the curriculum. (Lappan et. al., 1996, p. 38)

Technology is embedded in and used throughout the curriculum. Calculators and computers are used both in computation and in creating and manipulating representations. On many problems, students are asked to use their calculators to make tables and/or construct graphs. CMP problems then ask students to "explain", "interpret", "predict", and "compare" these representations.

For example, in a typical problem from the Moving Straight Ahead unit - ACE #4, p. 25 (Lappan et. al., 1998), students are given a table of values showing the time in hours and the distance in miles for one day's travel of a student's bike trip (see Figure 3). The problem asks students to generate a symbolic equation from the table of values (ostensibly by hand, but perhaps enterprising students could use the technology) and then to sketch a graph of the equation (using the calculator). The remaining parts of the problem ask students to answer various questions about the symbolic, graphical, or tabular representations. This problem is typical of CMP in that students are not instructed whether or not to use the technology. It is expected and assumed that technology will be used whenever a student feels that it might be useful in thinking about or completing a problem.

** Insert Figure 3 about here **

Elements in Typical Lessons

Finally, Algebra I lessons typically follow a fixed sequence of activities: Review the homework, present the new content, provide time for practicing the new material and perhaps some additional time for students to start their homework. Indeed, dividing the curriculum into small packages of new content—typically one new solution procedure or manipulation per lesson—generates its structure and sequence. This instructional format is closely related to the phases of Direct Instruction: (a) Introduction and review; (b) presentation; (c) guided practice; and (d) independent practice (Rosenshine, 1979). Direct instruction is a highly teacher-centered form of instruction and is an effective strategy for teaching mathematical procedures (Eggen and Kauchak, 1997).

In comparison, it is more difficult to characterize typical CMP lessons. "Investigations," which typically take more than one day to complete, are the smallest unit of curricular organization. Investigations are generally structured into three main phases: launch, explore, and summarize. In the "launch" phase, a problem context is clarified and established and work expectations are communicated. In the "explore" phase, students work to solve problem(s). In the "summarize" phase, students look for connections, patterns, and relationships in their own thinking and the mathematical content. Within each of these phases, however, daily lessons can be structured in quite different ways. Each phase can exhibit a mix of teacher presentation, small group (2-5 students) work, and whole class discussion. This potential mix of instructional formats means that the content and sequence of activities in consecutive days' lessons can be quite different. This form of instruction appears considerably more student-centered. Through exploration, analysis, and discussion of problems and solutions with the guidance of a supportive teacher, students can gain understanding of mathematical concepts.

Conclusions

Our efforts to identify key differences between older and newer conceptions of algebra was motivated by our desire to adequately assess students' understanding of algebra in the 8th grade. We feel that the six dimensions outlined above are a worthy (if incomplete) step toward that goal. In particular, we think Table 1 provides a much more productive basis for discussions and evaluations of algebra curricula than "new" vs. "old," "good" vs. "bad," or even "reform" vs. "traditional." Though we have illustrated the dimensions in comparing the CMP algebra curriculum with the content of Algebra I, Table 1 is more broadly applicable to other curricula that introduce students to algebra in middle and high school. Simply asking the questions associated with each of the dimensions, e.g., "what are the fundamental objects of study?," can be very useful, irrespective of the particular algebra curriculum in question.

But we also recognize that Table 1 lists features that <u>we</u> noticed when we examined and compared different introductions to algebra. Our experience with Algebra I and with newer approaches such as CMP, both as students and as teachers, has sensitized us to these differences. Given the current heated debates about algebra, many parents, teachers, and college professors are also cognizant of these same differences. But what about our students? They come to these curricula with much less experience of algebra but with expectations about the structure of daily lessons, the role of practice, the nature of an "answer," and other issues raised in Table 1. Moreover, they may find themselves moving between equation-based and functional relations-based algebra curricula from middle school to high school to college. What features do they see as different, and how they adjust to changes when they occur? Does the change in emphasis from equations and unknowns to functional relationships and variables register? Is the move into (or away from) the emphasis on multiple representations significant? Do students think about the change from daily lessons with regular structure to those that are part of longer explorations? These are some of the questions we want to address as we continue to explore what really is “new” in new algebra curricula.

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Table 1

Some key dimensions of differences between traditional Algebra I and 8th grade CMP

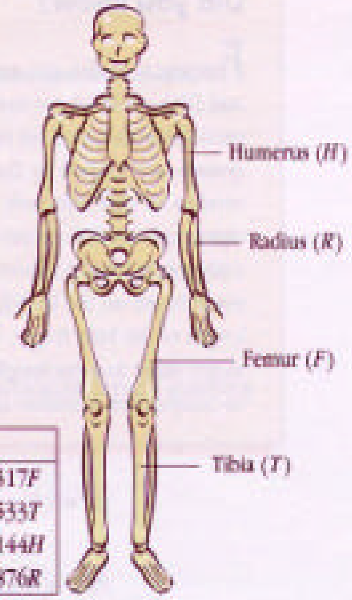
Algebra I	8th grade CMP
The fundamental objects in the curriculum	
Equations & symbolic expressions	Functional relationships represented in tables, graphs, and equations
Typical problems in the curriculum	
"Solve," "factor," "multiply," symbolic expressions or verbal statements with request to find a numerical value (word problems)	Verbal statements with tables, graphs, or symbolic expressions with request to find values and describe, explain, predict, etc.
Typical solution methods	
Complete the correct steps in symbolic procedures in the correct order	Relate verbal statements to tables, graphs, or equations; Compute or manipulate that representation; Interpret the results verbally
The role of practice	
Significant practice on particular problem types (in class and homework)	Similarities between problems are less salient; extended work on fewer, more open problems (in class and homework)
The role for technology for representing and calculating	
Used in balance with pencil & paper computation, which is more highly valued	Supports students' work on most all problems
Elements in a typical lesson	
Review homework, present new content, provide time for work on next assignment	More variation across lessons; Some mix of teacher presentation, small group work, and whole group discussion

Figure 1
Moving Straight Ahead, Investigation 4.3, p. 57

4.3 Analyzing Bones

Forensic scientists can estimate a person's height by measuring the length of certain bones, including the femur, the tibia, the humerus, and the radius.

The table below gives equations for the relationships between the length of each bone and the height for males and females. These relationships were found by scientists after much study and data collection. In the table, F represents the length of the femur, T the length of the tibia, H the length of the humerus, R the length of the radius, and h the person's height. All measurements are in centimeters.



Bone	Male	Female
Femur	$h = 69.089 + 2.238F$	$h = 61.412 + 2.317F$
Tibia	$h = 81.688 + 2.392T$	$h = 72.572 + 2.533T$
Humerus	$h = 73.570 + 2.970H$	$h = 64.977 + 3.144H$
Radius	$h = 80.405 + 3.650R$	$h = 73.502 + 3.876R$

Source: George Knill, "Mathematics in Forensic Science," *Mathematics Teacher* (February 1981): 31–32.

Problem 4.3

Use the equations on page 57 to answer parts A–D.

A. How tall is a female if her femur is 46.2 centimeters long?

B. How tall is a male if his tibia is 50.1 centimeters long?

C. If a woman is 152 centimeters (about 5 feet) tall, how long is her femur? Her tibia? Her humerus? Her radius?

D. If a man is 183 centimeters (about 6 feet) tall, how long is his femur? His tibia? His humerus? His radius?

Problem 4.3 Follow-Up

For one of the bones discussed above, graph the equations for males and females on the same set of axes. What do the x- and y-intercepts represent in terms of this problem? Does this make sense? Why?

Investigation 4: Solving Equations 57

Figure 2
Moving Straight Ahead, ACE #4-5, p. 10

Connections

4. In *Variables and Patterns*, you saw that the distance traveled by the tour van depended on time. Suppose the van averaged a steady 60 miles per hour on the interstate highway. The table below shows the relationship between the time traveled and the distance.

Time (hours)	0.5	1.0	1.5	2.0	2.5	3.0	3.5
Distance (miles)	30	60					

a. Copy and complete the table.

b. Make a coordinate graph of the data in the table.

c. Write a rule that describes the relationship between distance and time.

d. Predict the distance traveled in 8 hours.

e. Predict the time needed to travel 300 miles.

f. Pick a pair of (time, distance) values from the table. How is the pair related to the graph and the rule?

5. The soccer boosters make $5 on each T-shirt they sell. This can be described by the equation $A = 5n$, where A is the amount of money made and n is the number of T-shirts sold.

a. Make a table and a graph showing the amount of money made by selling up to ten T-shirts.

b. Compare the table and the graph from part a with the table and the graph you made for your experiment in Problem 1.1A or 1.1B. How are the tables similar? How are they different? How are the graphs similar? How are they different? What do you think causes the similarities and differences?

c. Compare the table, graph, and rule for the T-shirt sale with the table, graph, and rule in question 4. Describe the similarities and differences.

Figure 3
Moving Straight Ahead, ACE #4, p. 25

4. Mike was on the bike trip with José, Mario, and Melanie (from questions 1–3). He made the following table of the distances he traveled during day 1 of the trip.

Time (hours)	Distance (miles)
0	0
1	6.5
2	13
3	19.5
4	26
5	32.5
6	39

a. Assume Mike continued riding at this rate for the entire bike trip. Write an equation for the distance Mike traveled after t hours.

b. Sketch a graph of the equation.

c. When you made your graph, how did you choose the range of values for the time axis? For the distance axis?

d. How can you find the distance Mike traveled in 7 hours and in $9\frac{1}{2}$ hours, using the table? The graph? The equation?

e. How can you find the number of hours it took Mike to travel 100 miles and 237 miles, using the table? The graph? The equation?

f. For parts d and e, give the advantages and disadvantages of using each form of representation—a table, a graph, and an equation—to find the answers.

g. Compare the rate at which Mike rides with the rates at which José, Mario, and Melanie ride. Who rides the fastest? How can you determine this from the tables? From the graphs? From the equations?